Cauchy for Schubert

$$\sum_{w \in S_n} \mathfrak{S}_w(x)\,\mathfrak{S}_{w_0 w}(y) = \prod_{\substack{(i,j) \geq 1 \\ i+j \leq n}} (x_i + y_j)$$

More generally

<u>Thm</u>: $\mathfrak{S}_w(x;-y) = \sum_{\substack{u,v \in S_n \\ w = v^{-1}u \\ \ell(w) = \ell(v) + \ell(u)}} \mathfrak{S}_u(x)\,\mathfrak{S}_v(y)$

<u>Specializations</u>:

① $w = w_0$

② $w = \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & \cdots & n \\ n-k+1 & & \cdots & n & 1 & \cdots & n-k \end{pmatrix}$

$$\mathfrak{S}_w(x;-y) = \prod_{\substack{i=1,\dots,k \\ j=1,\dots,n-k}} (x_i + y_j)$$

Corollary: (dual Cauchy for Schur)

$$\sum_{\lambda\subseteq k\times(n-k)} S_\lambda(x_1,\dots,x_k)\, S_{\lambda'}(y_1,\dots,y_{n-k})$$

Cauchy formula follows from

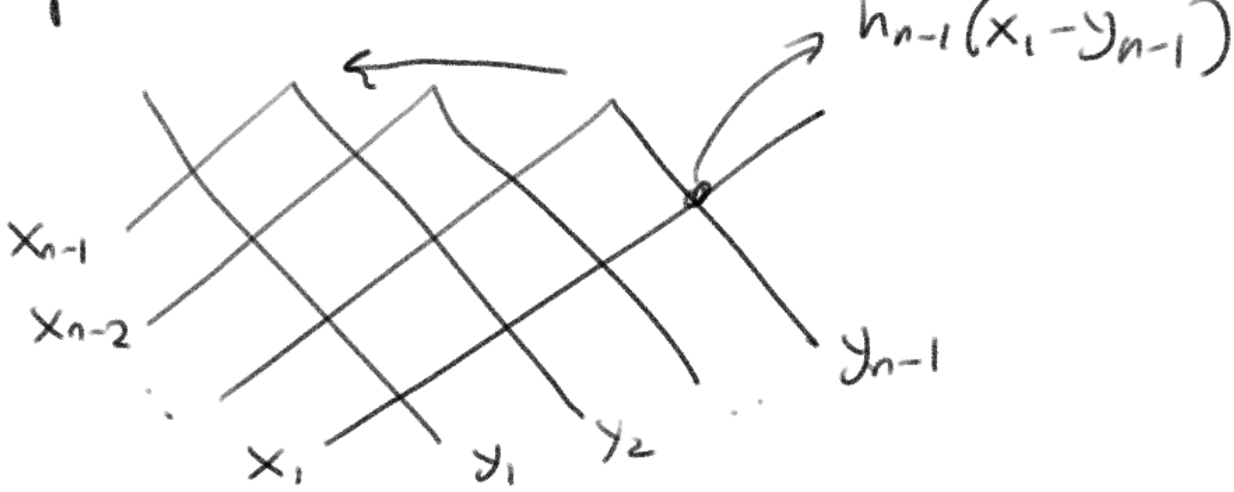
Thm $S(x;y) = S(0;y)\cdot S(x,0)$

where

$$S^{(n)}(x,y) = \prod_{i=1}^{n-1}\prod_{j=n-i}^{1} h_{i+j-1}(x_i-y_j)$$

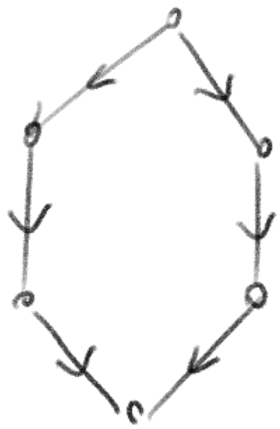
where $h_i(x) = 1 + x u_i$ satisfy YB relations

and the product is taken



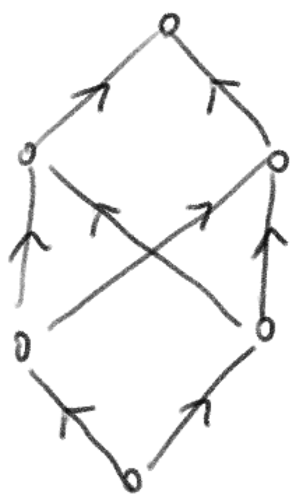
Monk–Chevellay Formula

- divided differences: top to bottom formula



weak Bruhat order

- monk–Chevellay: bottom to top formula



strong Bruhat

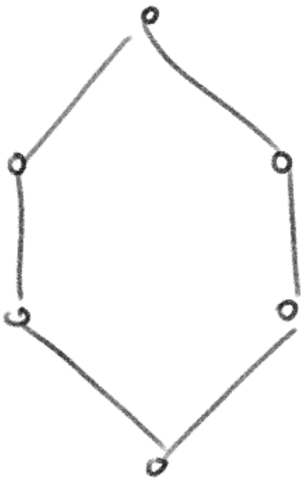
Weak vs Strong Bruhat Order

weak:

$u \lessdot w$ if

$w = u s_i$ &

$\ell(w) = \ell(u) + 1$

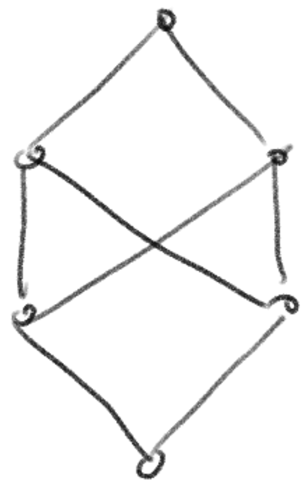


strong

$u \lessdot w$

$w = u t_{ij}$

$\ell(w) = \ell(u) + 1$



Thm $\forall\, w \in S_n$ and $i \in [n]$

$$x_i \cdot \mathfrak{S}_w \equiv \sum_{\substack{j>i \\ \ell(wt_{ij}) = \ell(w)+1}} \mathfrak{S}_{wt_{ij}} \;-\; \sum_{\substack{j'<i' \\ \ell(wt_{ij'}) = \ell(w)+1}} \mathfrak{S}_{wt_{ij}}$$

modulo $I_n = \langle e_1, \dots, e_n \rangle$ where

$e_i = e_i(x_1, \dots, x_n)$ elem. symm. polys.

Theorem $$\mathfrak{S}_{s_i} \cdot \mathfrak{S}_w \equiv \sum_{\substack{(a,b) \\ 1 \le a \le i < b \le n \\ \ell(w t_{ab}) = \ell(w)+1}} \mathfrak{S}_{w t_{ab}} \pmod{I_n}$$